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A Geometric Analysis of the Effects of Noise on Berry Phase

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Abstract In this work we describe the effect of classical and quantum noise on the Berry phase. It is not a topical review article but rather an overview of our work in this field aiming at giving a simple pictorial intuition of our results.

Keywords Berry phase · Noise · Fault tolerant quantum computation

1 A Basic Introduction to the Berry Phase

The first seminal paper [1] which described the implementation of an NMR two qubit quantum gate based on geometrical phase has triggered a considerable theoretical and experimental interest. The same scheme has been adapted for an implementation based on coupled superconducting nanostructures [2, 3], while shortly afterward a scheme for all geometrical universal quantum gates based on non abelian phases in coupled trapped ions has been put forward [4]. Since then several other experimental proposals have been discussed in literature. Such interest is largely due to the belief that geometric quantum gates should be intrinsically fault tolerant i.e. intrinsically robust against random noise—classical or quantum—of the experimental control parameters of the gate. In a nutshell the basic idea is that, being the Berry phase by its very nature "geometrical", i.e. dependent solely on the geometry of the path in the control parameters space, it should be resistant to random fluctuations of such path. Such appealing intuitive idea has been proved to be basically correct by a quantitative analysis. Here we will review in as much as possible pictorial terms our work on the effects of classical and quantum noise on the Berry phase.

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For the sake of completeness let us start by reviewing the basic concepts of geometric phase [5, 6]. Consider a Hamiltonian H depending on a set of control parameters which we generically indicate with the vector $\mathbf{r}$. If $\mathbf{r}$ changes in time slowly enough the energy eigenstates follow adiabatically the Hamiltonian. Let $|e_k(\mathbf{r})\rangle$ be the instantaneous energy eigenstates defined by the eigenvalue equation

$$H(\mathbf{r})|e_k(\mathbf{r})\rangle = E_k(\mathbf{r})|e_k(\mathbf{r})\rangle \tag{1}$$

and assume that the adiabatic time evolution of H is cyclical in a time T i.e. that $\mathbf{r}$ varies along a closed loop in the parameters space so that $\mathbf{r}(0) = \mathbf{r}(T)$. It was shown by Berry [5] that a system initially prepared in an energy eigenstate $|\psi(0)\rangle = |e_k(0)\rangle$ evolves at time T in a state which differs from the eigenstates at time 0 by a phase factor which is the sum of the usual dynamical phase Δ and a purely geometric phase Γ, i.e.

$$|\psi(T)\rangle = \exp(-i\Delta_k)\exp(i\Gamma_k)|e_k(0)\rangle \tag{2}$$

The dynamical phase (we have set $\hbar = 1$)

$$\Delta_k = \int_0^T E_k(\mathbf{r}(t))dt \tag{3}$$

gives information on how fast the parameter $\mathbf{r}(t)$ varies along the closed loop in parameter space while the geometric phase, expressed in terms of the so called Berry connection

$$\mathbf{A}_k = i\langle e_k(\mathbf{r})|\nabla_{\mathbf{r}}|e_k(\mathbf{r})\rangle \tag{4}$$

as

$$\Gamma_k = \oint \mathbf{A}_k \cdot d\mathbf{r} \tag{5}$$

depends only on the path followed by $\mathbf{r}$ i.e. it is independent on how fast $\mathbf{r}(t)$ changes in time.

To discuss a specific system let us consider a spin 1/2 in the presence of an external slowly varying magnetic field $\mathbf{B}(t) = B_0(t)\hat{\mathbf{n}}(t)$ with the unit vector $\hat{\mathbf{n}} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$. The Hamiltonian of such system, in appropriate units, takes the form

$$H(t) = \frac{1}{2}\mathbf{B}(t)\cdot\boldsymbol{\sigma} = \frac{1}{2}B_0(t)\hat{\mathbf{n}}\cdot\boldsymbol{\sigma} \tag{6}$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, σ_i are the Pauli operators. The classical field $\mathbf{B}(t)$ acts as an external control parameter, as its direction and magnitude can be experimentally changed. When varied adiabatically the instantaneous energy eigenstates follow the direction of $\hat{\mathbf{n}}$ and therefore can be expressed as

$$\begin{aligned} |\uparrow_n\rangle &= e^{-i\varphi/2}\cos\frac{\vartheta}{2}|\uparrow\rangle + e^{i\varphi/2}\sin\frac{\vartheta}{2}|\downarrow\rangle \\ |\downarrow_n\rangle &= e^{-i\varphi/2}\sin\frac{\vartheta}{2}|\uparrow\rangle - e^{i\varphi/2}\cos\frac{\vartheta}{2}|\downarrow\rangle \end{aligned} \tag{7}$$

where $|\uparrow\rangle, |\downarrow\rangle$ are the eigenstates of the σ_z operator. In this specific example therefore $\mathbf{r} = (\vartheta, \varphi)$.

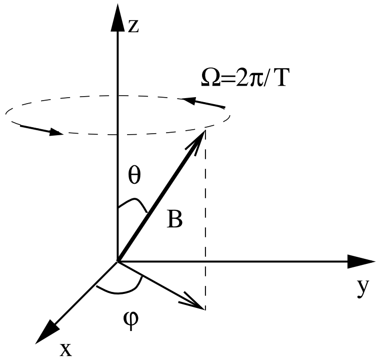


Fig. 1 A magnetic field **B** slowly precessing around the $\hat{\mathbf{z}}$ axis with angular velocity Ω

Suppose now that such adiabatic time evolution is cyclic i.e. that after a time T we have $\mathbf{B}(T) = \mathbf{B}(0)$, see Fig. 1. In this case the dynamical phase difference between energy eigenstates is

$$\Delta = 2\int_0^T B_0(t)dt \tag{8}$$

and depends only on the magnitude B_0 of the magnetic field but is independent on its direction $\hat{\mathbf{n}}(t)$. On the other hand since the eigenstates depend only on $\hat{\mathbf{n}}(t)$ the Berry phase depends only on ϑ, φ, how it is straightforward to verify calculating the components of **A**:

$$A_{\uparrow\varphi} = -A_{\downarrow\varphi} = i\langle\uparrow_n|\,\partial/\partial\varphi\,|\uparrow_n\rangle = \frac{1}{2}\cos\vartheta \tag{9}$$

$$A_{\uparrow\vartheta} = -A_{\downarrow\vartheta} = i\langle\uparrow_n|\,\partial/\partial\vartheta\,|\uparrow_n\rangle = 0 \tag{10}$$

To conclude this introductory section, let us consider the standard example of a slow precession of **B** at an angle ϑ around the z axis with angular velocity $\Omega = 2\pi/T \ll B_0$. A straightforward calculation shows that

$$\Gamma_\uparrow = -\Gamma_\downarrow = \int_0^{2\pi} A_{\uparrow\varphi}d\varphi = \pi\cos\vartheta \tag{11}$$

Note that the difference between the Berry phases $\Gamma_\uparrow - \Gamma_\downarrow$ is equal, modulo 2π, to the solid angle subtended by **B** with respect to the degeneracy $\mathbf{B} = 0$. This can be shown to be true not simply for the case of an adiabatic precession but in general for any closed loop in the $\{\vartheta, \varphi\}$ space.

2 The Berry Phase in the Presence of Classical Noise

We are now in the position to extend our analysis to the case in which the magnetic field contains a fluctuating component [7]. In this case the Hamiltonian (6) is modified as follows

$$H(t) = \frac{1}{2}\mathbf{B}_T\cdot\boldsymbol{\sigma} = \frac{1}{2}(\mathbf{B}(t) + \mathbf{F}(t))\cdot\boldsymbol{\sigma} \tag{12}$$

where we have divided the total magnetic field $\mathbf{B}_T$ into an average component **B** experimentally under our control and a fluctuating field **F**, Fig. 2. We will analyze the case in which **B**

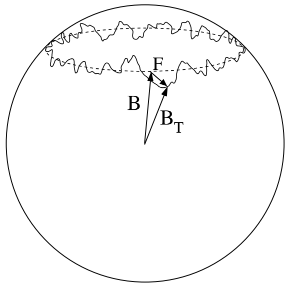


Fig. 2 The total magnetic field $\mathbf{B}_T$ is the sum of the vector $\mathbf{B}$ precessing around the $\hat{\mathbf{z}}$ axis and a fluctuating field $\mathbf{F}$

is a field of constant amplitude which undergoes a cyclic evolution while the components of $\mathbf{F}$ are random processes with zero average and small amplitude compared to $\mathbf{B}$, in order to consider lowest order corrections. Finally we will make the non unphysical assumption that the fluctuations are characterized by timescales such that the adiabatic approximation holds.

Corrections to (11) have a twofold origin as the fluctuating field $\mathbf{F}$ modifies both the connection $\mathbf{A}$ and the path. Up to terms $O(\delta\vartheta)$ the connection is

$$A_\varphi(\vartheta) \cong A_\varphi(\vartheta_0) + \frac{\partial A_\varphi}{\partial \vartheta}\delta\vartheta \tag{13}$$

$$= \frac{1}{2}(\cos\vartheta_0 - \delta\vartheta \sin\vartheta_0) \tag{14}$$

where ϑ_0 is the polar angle of the average field $\mathbf{B}$ while $\delta\vartheta$ is the fluctuation of the polar angle due to the fluctuating field $\mathbf{F}$. In order to analyze the corrections to the path, with no loss of generality, we again consider the case of a slow precession of $\mathbf{B}$ around the $\hat{\mathbf{z}}$ axis. In the presence of $\mathbf{F}$ the line element $d\mathbf{r}$ will have also a component perpendicular to the orbit of $\mathbf{B}$. However, as the connection $\mathbf{A}$ has zero component in the ϑ direction, we can restrict our attention to the φ component of $d\mathbf{r}$. To this end we write

$$d\mathbf{r} = \dot{\varphi} dt \cong (\dot{\varphi}_0 + \delta\dot{\varphi})dt \tag{15}$$

In (15) $\dot{\varphi}_0$ is the average angular velocity (in our case $\dot{\varphi}_0 = 2\pi/T$) while $\delta\dot{\varphi}$ is the first order correction due to $\mathbf{F}$. In particular when $\mathbf{F}$ fluctuates in the same direction of $\dot{\mathbf{B}}$ the precession speed increases while it decreases in the case of fluctuation in the opposite direction. Note that fluctuations in the path contain only corrections in φ while fluctuations in the connection depend only on ϑ. This is independent from any approximation but is due to the structure of the connection.

We can now express the Berry phase in the presence of noise as

$$\begin{aligned}\Gamma &= \int_0^T (A_\varphi(\vartheta_0) + \delta A_\varphi)(\dot{\varphi}_0 + \delta\dot{\varphi})dt \\ &\cong \Gamma_0 + \frac{2\pi}{T}\int_0^T \delta A_\varphi dt + A_\varphi(\vartheta_0)\int_0^T \delta\dot{\varphi} dt \\ &= \Gamma_0 - \frac{\pi}{T}\int_0^T \sin\vartheta_0 \delta\vartheta dt + A_\varphi(\vartheta_0)\delta\varphi(T)\end{aligned} \tag{16}$$

where the average Berry phase Γ_0 coincides with Γ in the absence of noise and it has been assumed for simplicity $\delta\varphi(0) = 0$. The last term in (16) is a non-cyclic contribution which

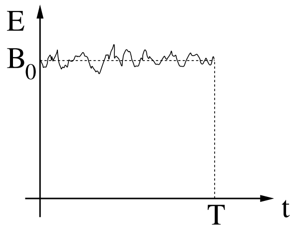


Fig. 3 The dynamical phase Δ, proportional to the time integral of the energy, in the presence of fluctuations becomes a random variable

appears when, due to the presence of **F**, $\mathbf{B}_T$ does not return to its original. In this case, instead of the geometrical phase definition given by Berry, which assumes that the Hamiltonian is periodic, we have to use the definition by Samuel and Bhandari [8] relative to non cyclic evolution. If this is done the third term does not appear and (16) becomes:

$$\Gamma = \Gamma_0 - \frac{\pi}{T}\int_0^T \sin\vartheta_0 \delta\vartheta\, dt \tag{17}$$

In order to proceed a physical model for the noise is needed, in other words a stochastic process for **F** must be assigned. Given the probability distribution for the field it is straightforward to calculate the distribution for the Berry phase. To this end it is convenient to express the trigonometric functions appearing in (17) in terms of the fluctuating field components F_i. Let ϑ_0, φ_0 be the polar angles of **B**; ϑ, φ those of $\mathbf{B}_T$; and $\delta\vartheta$, $\delta\varphi$ the first order differences between the polar angles of the two fields. If we expand in Taylor series $\cos\vartheta$ we obtain:

$$\cos(\vartheta_0 + \delta\vartheta) \cong \cos\vartheta_0 - \delta\vartheta \sin\vartheta_0 = \frac{B_z}{B_0} + \frac{F_z}{B_0} - \frac{B_z}{B_0^3}\mathbf{B}\cdot\mathbf{F}$$

and therefore

$$-\delta\vartheta \sin\vartheta_0 = \frac{F_z}{B_0} - \frac{B_z}{B_0^3}\mathbf{B}\cdot\mathbf{F} \tag{18}$$

Substituting (18) in (17) we find:

$$\Gamma = \Gamma_0 + \frac{\pi}{T}\int_0^T \left[\frac{F_z}{B_0} - \frac{B_z}{B_0^3}\mathbf{B}\cdot\mathbf{F}\right] dt \tag{19}$$

Until now we concentrated only in the geometrical phase; however during an adiabatic cyclic evolution the eigenstates acquire both the dynamical and geometrical phase. It is known that the dynamical phase Δ is proportional to the modulus of the magnetic field. This means that the dynamical phase becomes a stochastic processes like the Berry phase, see Fig. 3. We can write Δ in terms of the fields **B** and **F** as we did for the geometrical phase:

$$\Delta = \Delta_0 + \int_0^T \frac{\mathbf{B}\cdot\mathbf{F}}{B_0} dt = \Delta_0 + \int_0^T \hat{\mathbf{n}}\cdot\mathbf{F} dt \tag{20}$$

where $\Delta_0 = (B_0/2)T$. Let us now analyze the effect of noise on the coherence of a system. Suppose we prepare the system in a state which is a superposition of the two eigenstate of the Hamiltonian at time $t = 0$:

$$|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle \tag{21}$$

After a slow cyclic evolution the eigenstates have acquired both the dynamical and geometrical phases and the final state is:

$$|\psi'(\alpha)\rangle = ae^{i\alpha}|\uparrow\rangle + be^{-i\alpha}|\downarrow\rangle \tag{22}$$

where $\alpha = \Gamma + \Delta$ is the total phase. In the presence of noise this phase is a random variable with probability distribution $P(\alpha)$, therefore the system at the end of the evolution is in a mixed state described by a density operator which can be written as

$$\rho = \int |\psi'(\alpha)\rangle\langle\psi'(\alpha)|P(\alpha)d\alpha \tag{23}$$

If we insert (22) in the definition of ρ we find that the populations are unchanged while the coherences are shrunk. In terms of the Bloch vector, this means that the z component is unchanged while the component in the x–y plane is reduced, i.e. noise induces dephasing. It is worth pointing out that since both the dynamical and geometrical phases depend on **F**, they are not independent processes and $P(\alpha) \neq P(\Gamma_B)P(\Delta)$. From (17) and (20) it is possible to find the probability distribution for Γ and Δ once that for **F** is known. In [7] we have assumed that the components of the fluctuating field **F** are independent Ornstein–Uhlenbeck process, i.e. they are gaussian, stationary and markovian with a lorentzian spectrum with bandwidths much smaller than B_0. With these assumption we found that the distribution for Γ and Δ is a gaussian whose average value is the sum of the Berry and the dynamical phases in the absence of noise. However, as anticipated, the rate of decoherence induced by the noise on the geometric and geometric components of the overall phase are very different. Rather than re-deriving the exact expression for the variance of the overall probability distribution let us analyze in pictorial terms the origin of the different impact of noise on the dephasing rates of the dynamical and of the geometrical phases. The origin of such different impacts is the fact that while Γ comes from an integral in parameter space, Δ comes from an integral in the time domain. If we express Γ in terms of an integral in time we obtain

$$\Gamma_B = \int_0^{2\pi} A_\varphi d\varphi = \frac{2\pi}{T}\int_0^T A_\varphi \dot{\varphi} dt \tag{24}$$

If we compare (8) with (24) we see that the two differ by a factor $1/T$. Suppose we slow down the adiabatic precession of **B** e.g. let us suppose we double the period T. In this case the effect of energy fluctuations will add up as shown in Fig. 4a the consequence of this is that, as it is typical in diffusive processes, the contribution of energy fluctuations on the variance of the Gaussian distribution $P(\alpha)$ grows linearly in time. Doubling the precession period however has a different effect on the geometric contribution of the variance of $P(\alpha)$. Such contribution was shown in [7] to decreases as $1/T$. In pictorial terms—see Fig. 4b—this can be understood by noting that, the longer the precession time the "tighter" are the fluctuations on the path and the smaller the average square fluctuation on the solid angle subtended by **B**. A different behavior would have appeared if the fluctuation of **B** were dependent on ϑ, φ but not explicitly on time i.e. if the fluctuations were static. In this case the path fluctuations would be independent of the precession time T.

The ideal experimental situation i.e. the one less subject to decoherence, is the one in which **B** is a vector of constant modulus randomly jittering around the desired path, in other words the one in which the only fluctuating parameter is $\hat{\mathbf{n}}$ but not B_0. This would lead to no fluctuations on the dynamical phase and an effect on the geometric phase which decreases as $1/T$.

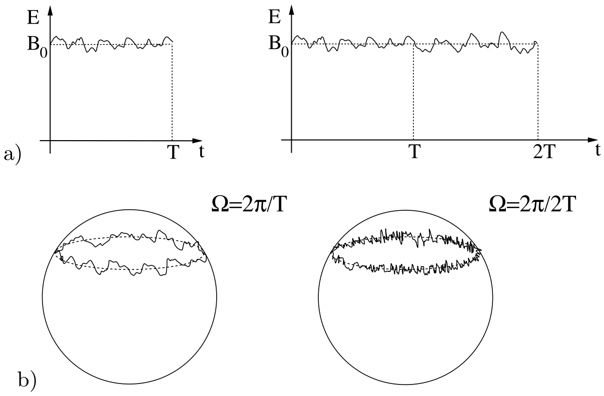


Fig. 4 The effect of doubling the period of the precession T has a different effect on the dynamic and geometric phase: **a** the first is integrated in the time domain; **b** the second is integrated in parameter space

3 The Correction to the Berry Phase in the Presence of Quantum Noise

In the previous section we have described the effects of classical fluctuations in the control parameters. In this section we will extend our analysis to the effects of quantum noise. Similar effects have been analyzed with the quantum jumps approach [9, 10] and with a diagrammatic technique which leads to an effective master equation [11, 12]. Not only we will see that our results on the decoherence time are confirmed but also we will see how transitions induced by quantum fluctuations modify the Berry phase [13]. To this end let us assume that our spin interacts not only with a slowly varying classical magnetic field but also with an environment modeled as a bath of harmonic oscillators. The overall system Hamiltonian is assumed to be of the standard form

$$H = \frac{1}{2}\mathbf{B}\cdot\boldsymbol{\sigma} + \sum_k \omega_k a_k^\dagger a_k + \sum_k g_k \sigma_z \left(a_k + a_k^\dagger\right) \tag{25}$$

where $a_k^\dagger (a_k)$ are bosonic creation (annihilation) operators for mode k. The effects of quantum noise are better analyzed by describing the system dynamics in terms of the so called adiabatic Hamiltonian [14, 15] i.e. of the Hamiltonian whose eigenstates, in the absence of interaction with the environment, after a cyclic evolution acquire the dynamical and geometrical phase predicted by Berry. To start with let us rewrite the free spin Hamiltonian in the form

$$\hat{H}_S \equiv \frac{1}{2}\mathbf{B}\cdot\boldsymbol{\sigma} = \frac{B_0}{2}\left(|{\uparrow_n}(t)\rangle\langle{\uparrow_n}(t)| - |{\downarrow_n}(t)\rangle\langle{\downarrow_n}(t)|\right) \tag{26}$$

and let us define the following time dependent unitary operator:

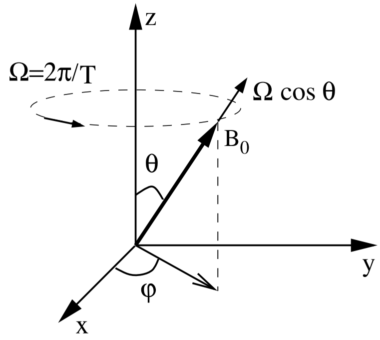
$$U(t) = |{\uparrow_n}(0)\rangle\langle{\uparrow_n}(t)| + |{\downarrow_n}(0)\rangle\langle{\downarrow_n}(t)| \tag{27}$$

In the absence of any coupling with the environment the time evolution of the state vector $|\tilde{\psi}(t)\rangle = U(t)|\psi(t)\rangle$ is generated by the Hamiltonian

$$\hat{\tilde{H}}_S = U(t)\hat{H}_S U^\dagger(t) - iU(t)\frac{d}{dt}U^\dagger(t) \tag{28}$$

When the Hamiltonian varies slowly enough i.e. as long as $\langle\uparrow(t)|\frac{d}{dt}|\downarrow(t)\rangle \ll B_0$, we can neglect the transitions between the instantaneous energy states $|{\uparrow_n}(t)\rangle$ and $|{\downarrow_n}(t)\rangle$. This

Fig. 5 In the new reference frame the magnetic field increases in modulus by $\Omega\cos\vartheta$ but its direction does not change



is nothing but the standard adiabatic approximation [16], and amounts to assume $\langle\uparrow(t)|\times\frac{d}{dt}|\downarrow(t)\rangle=0$. The adiabatic Hamiltonian is therefore

$$\hat{H}_S^{ad}=\left(\frac{B_0}{2}-i\langle\uparrow_n(t)|\frac{d}{dt}|\uparrow_n(t)\rangle\right)|\uparrow_n(0)\rangle\langle\uparrow_n(0)| + \left(-\frac{B_0}{2}-i\langle\downarrow_n(t)|\frac{d}{dt}|\downarrow_n(t)\rangle\right)|\downarrow_n(0)\rangle\langle\downarrow_n(0)| \tag{29}$$

Note that when **B** undergoes a cyclic evolution the eigenstates of (29) correctly acquire the dynamical plus the geometrical phase predicted by Berry. From (7) it follows that

$$i\langle\uparrow_n(t)|\frac{\partial}{\partial t}|\uparrow_n(t)\rangle=-i\langle\downarrow_n(t)|\frac{\partial}{\partial t}|\downarrow_n(t)\rangle=\dot{\varphi}\frac{1}{2}\cos\vartheta \tag{30}$$

The adiabatic Hamiltonian takes therefore the form

$$\hat{H}_S^{ad}=\frac{B_0+\dot{\varphi}\cos\vartheta(t)}{2}\hat{\mathbf{n}}(0)\cdot\boldsymbol{\sigma} \tag{31}$$

For the sake of simplicity, and with no loss of generality, we will restrict again our analysis of a slow precession of $\mathbf{B}(t)$ with angular velocity Ω i.e. $\varphi(t)=\Omega t, \varphi(0)=0, \vartheta(t)=\vartheta(0)$. Furthermore, if we rotate frame so that $[\cos\vartheta(0)\sigma_z+\sin\vartheta(0)\sigma_x]\to\sigma_z$ the adiabatic Hamiltonian takes the following simple form

$$H_S^{ad}=\frac{\omega_0}{2}\sigma_z \tag{32}$$

where $\omega_0=B_0+\Omega\cos\vartheta(0)$. It is important to note that our adiabatic Hamiltonian is not the spin Hamiltonian expressed in the rotating frame. In the standard rotating frame a field of amplitude Ω along the $\hat{\mathbf{z}}$ direction is added to **B**, i.e. in the rotating frame the effective magnetic field changes both in direction and length. This is not the case in our adiabatic Hamiltonian, in which the magnetic field changes only in length, see Fig. 5. This is a consequence of the fact that we assume from the very beginnings the adiabatic limit and we do not need to discard a posteriori the non adiabatic terms. Our full, time independent, adiabatic spin-boson Hamiltonian takes therefore the form

$$H=\frac{\omega_0}{2}\sigma_z+\sum_k\omega(k)a_k^\dagger a_k+\sum_k g_k\left(a_k+a_k^\dagger\right)(\sigma_z\cos\vartheta-\sigma_x\sin\vartheta) \tag{33}$$

In [13] the above Hamiltonian was used to derive a set of quantum Langevin equations of motion which lead in a natural way to frequency shifts, corrections to the geometric phases and damping constants. Here we will describe the obtained results providing a simple physical and geometrical picture. Let us first note that the quantum noise has a component "parallel" to the z direction i.e. coupled to σ_z. Such noise is responsible of pure dephasing but does not induce transitions between the energy eigenstates. It is immediate to convince oneself that such interaction term does not have any effect on the geometric or dynamic phase. The decay constant $\gamma_{\parallel}$ which describes such pure dephasing turns out to be

$$\gamma_{\parallel} = \pi \int_0^{\omega_c} d\omega_k \rho(\omega_k) g_k^2 \,(2n_k + 1)\, \delta(\omega_k) \tag{34}$$

where $\rho(\omega_k)$ is the density of modes at frequency ω_k and n_k is the mean number of photon in field mode k, and ω_c is a cutoff frequency. Note that $\gamma_{\parallel}$ is not modified by the adiabatic precession. Furthermore its value depends on the density of field modes at zero frequency which, in most situations of physical interest it is equal to zero.

The interaction hamiltonian (33) contains also a term proportional to σ_x which is responsible of the exchange of energy between system and bath. This exchange of energy has the twofold effect of inducing real and virtual transitions. Real transitions generate dissipation at a rate proportional to

$$\gamma_{\perp} = \pi \int_0^{\omega_c} d\omega_k \rho(\omega_k) g_k^2 \,(2n_k + 1)\, \delta(\omega_0 - \omega_k) \tag{35}$$

The decay constant $\gamma_{\perp}$ depends on the density of modes at the resonance frequency $\omega = \omega_0$. If we assume that the density of modes is a slowly varying function of ω near resonance, we can safely assume for very small Ω, i.e. in the adiabatic limit, $\int_0^{\omega_c} d\omega_k \rho(\omega_k) g_k^2 (2n_k + 1) \times \delta(\omega - \omega_0) \approx \int_0^{\omega_c} d\omega_k \rho(\omega_k) g_k^2 (2n_k + 1)\delta(\omega - B_0)$. This confirms that the timescale of dipole decay is not modified by the adiabatic evolution, a result which is in line the results discussed in the previous section and which has been obtained in the quantum case with different techniques, from quantum jump [9] to perturbative diagrammatic techniques [11]. We should point out that in order to observe the geometric phase we must have

$$\gamma_{\perp} \ll \Omega \ll \omega_0 \tag{36}$$

Virtual transitions, as well known, are responsible of a frequency shift which turns out to be

$$\lambda = \int_0^{\omega_c} d\omega_k \rho(\omega_k) g_k^2 (2n_k + 1) \left(\frac{P}{\omega_0 - \omega_k} + \frac{P}{\omega_0 + \omega_k} \right) \tag{37}$$

In the adiabatic limit we must consider terms up to order $O(\Omega)$ and therefore

$$\begin{aligned} \lambda &= \int_0^{\omega_c} d\omega_k \rho(\omega_k) g_k^2 (2n_k + 1) \\ &\quad \times \left[\left(\frac{P}{B_0 - \omega} + \frac{P}{B_0 + \omega} \right) + \Omega \cos\vartheta \, \frac{\partial}{\partial \omega_0} \bigg|_{\omega_0 = B_0} \left(\frac{P}{\omega_0 - \omega} + \frac{P}{\omega_0 + \omega} \right) \right] \\ &\approx \lambda_0 + \delta\lambda \end{aligned} \tag{38}$$

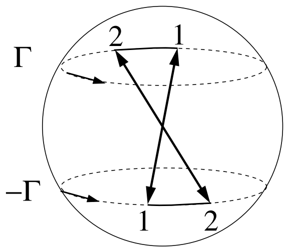


Fig. 6 Virtual transitions between orthogonal eigenstates not only are responsible for the Lamb shift but also give origin to a negative contribution to the average Berry phase. Such correction is proportional to the probability of a virtual transition

The quantity $\sin^2\vartheta\,\lambda_0$ is nothing but the Lamb Shift [17, 18], while

$$\delta\lambda = -\Omega\cos\vartheta\sum_k g_k^2(2n_k+1)\left[\frac{1}{(B_0-\omega)^2}+\frac{1}{(B_0+\omega)^2}\right] \tag{39}$$

gives information on the effect of the quantum fluctuations on the geometric phase. This correction coincides with the results obtained by [11] with an elaborated perturbative expansion. The observable overall phase difference between the two energy eigenstates at the end of their cyclic evolution, i.e. at time T, will be

$$\Phi(T) = 2\Delta + 2\Gamma \tag{40}$$

where the dynamical phase difference 2Δ

$$2\Delta = \left[B_0 + \sin^2\vartheta\int_0^{\omega_c} d\omega_k\rho(\omega_k)g_k^2(2n_k+1)\left(\frac{P}{B_0-\omega}+\frac{P}{B_0+\omega}\right)\right]T \tag{41}$$

is simply due to the renormalized energy splitting, while the geometric phase difference 2Γ is

$$2\Gamma = 2\pi\cos\vartheta\left\{1-\sin^2\vartheta\int_0^{\omega_c} d\omega_k\rho(\omega_k)g_k^2(2n_k+1)\left[\frac{1}{(B_0-\omega)^2}+\frac{1}{(B_0+\omega)^2}\right]\right\} \tag{42}$$

The expression (42) is amenable to a straightforward intuitive geometric interpretation, Fig. 6. As we have discussed in the previous section, for a spin $1/2$ the Berry phase is equal to the solid angle spanned by the time varying magnetic field **B** on a unit sphere centered around degeneracy. As opposite energy eigenstates acquire opposite geometric phases the overall phase difference between them will be, for a slowly precessing field at an angle ϑ, equal to $2\Gamma_0 = 2\pi\cos\vartheta$. In the presence of a weak coupling with the bosonic bath however each energy eigenstate will undergo virtual transitions, responsible for the Lamb Shift, with a probability Π_{vt} which, a straightforward second order perturbation theory shows to be

$$\Pi_{vt} = \sin^2\vartheta\int_0^{\omega_c} d\omega_k\rho(\omega_k)g_k^2(2n_k+1)\left[\frac{1}{(B_0-\omega)^2}+\frac{1}{(B_0+\omega)^2}\right] \tag{43}$$

During such transition the spin state parallel (antiparallel) to the direction of the field **B** 'jumps' to the antiparallel (parallel) spin state, acquiring an opposite geometric phase, as shown in Fig. 6. The overall geometric phase difference between the energy eigenstates will be therefore decreased by an amount proportional to Π_{vt}. Notice that the correction to the Berry phase is of order $O(g^2)$. In our discussion on the effects of classical noise no analogous correction was obtained because only contributions to first order in the fluctuating field were considered.

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